Note on "Vacuum stability of a general scalar potential of a few fields"*

Yisheng Song
School of Mathematical Sciences, Chongqing Normal University,
Chongqing, P.R. China, 401331; Email: yisheng.song@cqnu.edu.cn

Abstract

The purpose of this letter is to point out that some conclusions in the paper (Eur. Phys. J. C **76**, 324(2016)) are incomplete, and to give complete and improved conclusions. The analytic necessary and sufficient conditions are given for the boundedness-from-below conditions of general scalar potentials of two real scalar fields ϕ_1 and ϕ_2 and the Higgs bonson **H**.

Keyword: Positive definiteness; Homogeneous polynomial; Analytical expression.

1 Introduction

Kannike [1] presented the boundedness-from-below conditions of general scalar potentials of two real scalar fields ϕ_1 and ϕ_2 and the Higgs bonson **H**,

$$\begin{aligned} V(\phi_1,\phi_2,|H|) =&\lambda_H|H|^4+\lambda_{H20}|H|^2\phi_1^2+\lambda_{H11}|H|^2\phi_1\phi_2+\lambda_{H02}|H|^2\phi_2^2 \\ &+\lambda_{40}\phi_1^4+\lambda_{31}\phi_1^3\phi_2+\lambda_{22}\phi_1^2\phi_2^2+\lambda_{13}\phi_1\phi_2^3+\lambda_{04}\phi_2^4. \end{aligned} \tag{1}$$

This is equivalent to an analytic necessary and sufficient conditions of

$$V(\phi_1,\phi_2,|H|)>0 \text{ for all } \phi_1,\phi_2,\mathbf{H}.$$

However, there is a question in Kannike's conclusions. Eqs.(54) and (55) in Kannike [1] is inaccuracy.

For two real scalar fields ϕ_1 and ϕ_2 and the Higgs bonson **H**, a general scalar potentials $V(\phi_1,\phi_2,|H|)$ is rewritten as follows

$$V(\phi_1,\phi_2,|H|)=\lambda_H|H|^4+M^2(\phi_1,\phi_2)|H|^2+V(\phi_1,\phi_2),$$

*This work was supported by the National Natural Science Foundation of P.R. China (Grant No.12171064), by The team project of innovation leading talent in chongqing (No.CQYC20210309536) and by the Foundation of Chongqing Normal university (20XLB009)

where

$$M^2(\phi_1,\phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2,$$
$$V(\phi_1,\phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4. \tag{2}$$

So applying the well-known positivity conditions of quadratic polynomial

$$p(t) = at^2 + bt + c$$

for all $t = |H|^2 \geq 0$ (which is showed hundreds of years ago), $V(\phi_1,\phi_2,|H|) > 0$ for all $\phi_1,\phi_2,\mathbf{H}$ ($a = \lambda_H > 0$) if and only if for all ϕ_1,ϕ_2,

$$\begin{cases} b = M^2(\phi_1,\phi_2) \geq 0, c = V(\phi_1,\phi_2) > 0; \\ b = M^2(\phi_1,\phi_2) < 0, 4ac - b^2 = 4\lambda_H V(\phi_1,\phi_2) - (M^2(\phi_1,\phi_2))^2 > 0. \end{cases} \tag{3}$$

It is obvious that $M^2(\phi_1,\phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2$ is a quadric form with respect to two variables ϕ_1,ϕ_2, and hence, the inequality $M^2(\phi_1,\phi_2) \geq 0$ is equivalent to positive semi-definiteness of its coefficient matrix M^2. Then by Sylvester's criterion, $M^2(\phi_1,\phi_2) \geq 0$ if and only if

$$\lambda_{H20} \geq 0,\ \lambda_{H02} \geq 0,\ \lambda_{H20}\lambda_{H02} - \frac{1}{4}\lambda_{H11}^2 \geq 0. \tag{4}$$

The inequality $M^2(\phi_1,\phi_2) < 0$ is equivalent to negative definiteness of its coefficient matrix. That is, $-M^2(\phi_1,\phi_2) > 0$, i.e., the matrix

$$-M^2 = \begin{pmatrix} -\lambda_{H20} & -\frac{1}{2}\lambda_{H11} \\ -\frac{1}{2}\lambda_{H11} & -\lambda_{H02} \end{pmatrix}$$

is positive definite if and only if

$$\lambda_{H20} < 0,\ \lambda_{H02} < 0,\ \lambda_{H20}\lambda_{H02} - \frac{1}{4}\lambda_{H11}^2 > 0. \tag{5}$$

So, Eqs.(4) and (5) are differ from Eqs.(54) and (55) of Kannicke [1]. Moreover, the conclusion Eq.(68) of Kannicke [1] may not hold also.

2 Boundedness-from-below conditions

Now we correct this mistake and present the analytic necessary and sufficient conditions are showed for the boundedness from below of scalar potential of two real scalar fields ϕ_1 and ϕ_2 and the Higgs doublet $\mathbf{H}$. It follows from the conclusion (3) that we firstly need the analytic necessary and sufficient conditions of $V(\phi_1,\phi_2) > 0$,

$$V(\phi_1,\phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4. \tag{6}$$

It is obvious that the discriminant $D \geq 0$ is a necessary condition of $V(\phi_1, \phi_2) > 0$. Such a positivity condition may trace back to ones of Refs. Rees [2] , Lazard [3] Gadem-Li [4], Ku [5] and Jury-Mansour [6]. Untill to 2005, Wang-Qi [7] improved their proof, and perfectly gave analytic necessary and sufficient conditions. For more detail about applications of these results, see Song-Qi [8] also. That is, for all ϕ_1, ϕ_2 with $(\phi_1, \phi_2) \neq (0, 0)$, the binary quartic homogeneous polynomial (6), $V(\phi_1, \phi_2) > 0$ if and only if

$$\begin{cases} \lambda_{40} > 0, \lambda_{04} > 0, \\ D = 0,\ G = 0,\ R = 0 \text{ and } Q > 0; \\ D > 0 \text{ and } Q \geq 0, \text{ or } Q < 0 \text{ and } R > 0 \end{cases} \tag{7}$$

where

$$\begin{aligned} G =& \frac{1}{4}\lambda_{40}^2\lambda_{13} - \frac{1}{8}\lambda_{40}\lambda_{31}\lambda_{22} + \frac{1}{32}\lambda_{31}^3 \\ Q =& \frac{1}{6}\lambda_{40}\lambda_{22} - \frac{1}{16}\lambda_{31}^2 = \frac{1}{48}(8\lambda_{40}\lambda_{22} - 3\lambda_{31}^2) \\ I =& \lambda_{40}\lambda_{04} - \frac{1}{4}\lambda_{31}\lambda_{13} + \frac{1}{12}\lambda_{22}^2 \\ J =& \frac{1}{6}\lambda_{40}\lambda_{22}\lambda_{04} + \frac{1}{48}\lambda_{31}\lambda_{22}\lambda_{13} - \frac{1}{216}\lambda_{22}^3 \\ & - \frac{1}{16}\lambda_{40}\lambda_{13}^2 - \frac{1}{16}\lambda_{31}^2\lambda_{04} \\ D =& I^3 - 27J^2, R = \lambda_{40}^2 I - 12Q^2. \end{aligned}$$

Recently, Qi-Song-Zhang [9] gave a new necessary and sufficient condition other than the above results (7) in forms.

Next we give the revised version of the conclusion Eq.(68) in Kannicke [1]. Let $V'(\phi_1, \phi_2) = 4\lambda_H V(\phi_1, \phi_2) - (M^2(\phi_1, \phi_2))^2$. Now we show $V'(\phi_1, \phi_2) > 0$.

$$\begin{aligned} V'(\phi_1, \phi_2) =& 4\lambda_H V(\phi_1, \phi_2) - (M^2(\phi_1, \phi_2))^2 \\ =& (4\lambda_{40}\lambda_H - \lambda_{H20}^2)\phi_1^4 + (4\lambda_H\lambda_{31} - 2\lambda_{H20}\lambda_{H11})\phi_1^3\phi_2 \\ & + (4\lambda_H\lambda_{22} - 2\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2)\phi_1^2\phi_2^2 \\ & + (4\lambda_H\lambda_{13} - 2\lambda_{H02}\lambda_{H11})\phi_1\phi_2^3 + (4\lambda_{04}\lambda_H - \lambda_{H02}^2)\phi_2^4 \\ =& \lambda'_{40}\phi_1^4 + \lambda'_{31}\phi_1^3\phi_2 + \lambda'_{22}\phi_1^2\phi_2^2 + \lambda'_{13}\phi_1\phi_2^3 + \lambda'_{04}\phi_2^4, \end{aligned}$$

where

$$\begin{aligned} &\lambda'_{40} = 4\lambda_{40}\lambda_H - \lambda_{H20}^2,\ \lambda'_{04} = 4\lambda_{04}\lambda_H - \lambda_{H02}^2, \\ &\lambda'_{31} = 4\lambda_H\lambda_{31} - 2\lambda_{H20}\lambda_{H11},\ \lambda'_{13} = 4\lambda_H\lambda_{13} - 2\lambda_{H02}\lambda_{H11}, \\ &\lambda'_{22} = 4\lambda_H\lambda_{22} - 2\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2. \end{aligned}$$

In terms of the coefficients of $V'(\phi_1,\phi_2)$, we define the following quantities:

$$\begin{aligned}
G' =&\frac{1}{4}\lambda_{40}'^2\lambda_{13}' - \frac{1}{8}\lambda_{40}'\lambda_{31}'\lambda_{22}' + \frac{1}{32}\lambda_{31}'^3\\
Q' =&\frac{1}{6}\lambda_{40}'\lambda_{22}' - \frac{1}{16}\lambda_{31}'^2\\
I' =&\lambda_{40}'\lambda_{04}' - \frac{1}{4}\lambda_{31}'\lambda_{13}' + \frac{1}{12}\lambda_{22}'^2\\
J' =&\frac{1}{6}\lambda_{40}'\lambda_{22}'\lambda_{04}' + \frac{1}{48}\lambda_{31}'\lambda_{22}'\lambda_{13}' - \frac{1}{216}\lambda_{22}'^3\\
&- \frac{1}{16}\lambda_{40}'\lambda_{13}'^2 - \frac{1}{16}\lambda_{31}'^2\lambda_{04}'\\
D' =&I'^3 - 27J'^2, R' = \lambda_{40}'^2 I' - 12Q'^2.
\end{aligned}$$

Then an application of the conclusion (7), we have $V'(\phi_1,\phi_2) > 0$ for all ϕ_1,ϕ_2 with $(\phi_1,\phi_2) \neq (0,0)$ if and only if

$$\begin{cases}\lambda_{40}' > 0, \lambda_{04}' > 0,\\ D' = 0,\ G' = 0,\ R' = 0 \text{ and } Q' > 0;\\ D' > 0 \text{ and } Q' \geq 0 \text{ or } Q' < 0 \text{ and } R' > 0.\end{cases} \tag{8}$$

Altogether, combing Eq. (3) and Eqs. (7), (4), (5), (8), the analytic necessary and sufficient condition is established for the boundedness from below of scalar potential of two real scalar fields ϕ_1 and ϕ_2 and the Higgs doublet $\mathbf{H}$. That is, $V(\phi_1,\phi_2,|\mathbf{H}|) > 0$ for all $\phi_1,\phi_2,\mathbf{H}$ with $(\phi_1,\phi_2,\mathbf{H}) \neq (0,0,0)$ if and only if

$$\begin{cases}\lambda_H > 0, \lambda_{40} > 0, \lambda_{04} > 0 \text{ and}\\ (i)\ \lambda_{H20} \geq 0,\ \lambda_{H02} \geq 0,\ 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 \geq 0,\\ \quad D = 0,\ G = 0,\ R = 0 \text{ and } Q > 0;\\ \quad D > 0 \text{ and either } Q \geq 0, \text{ or } Q < 0 \text{ and } R > 0;\\ (ii)\ \lambda_{H20} < 0,\ \lambda_{H02} < 0,\ 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 > 0,\\ \quad 4\lambda_{40}\lambda_H - \lambda_{H20}^2 > 0,\ 4\lambda_{04}\lambda_H - \lambda_{H02}^2 > 0 \text{ and}\\ \quad D' = 0,\ G' = 0,\ R' = 0 \text{ and } Q' > 0;\\ \quad D' > 0 \text{ and } Q' \geq 0, \text{ or } Q' < 0 \text{ and } R' > 0.\end{cases} \tag{9}$$

References

[1] K. Kannike, Eur. Phys. J. C **76**, 324(2016); Erratum: Eur. Phys. J. C **78**, 355(2018)

[2] E. L. Rees, Amer. Math. Monthly, **29(2)**, (1922) 51-55

[3] D. Lazard, J. Sym. Comput. **5(1-2)**, (1988) 261-266

[4] R.N. Gadem, C.C. Li, IEEE Trans. Autom. Control. **AC-9**, (1964) 187-188

[5] W.H. Ku, IEEE Tram. Autom. Control. **AC-10(3)**, (1965) 372-373

[6] E.I. Jury, M. Mansour, IEEE Trans. Autom. Control. **AC-26(2)**, (1981) 444-451

[7] F. Wang, L. Qi, IEEE Trans. Autom. Control. **50(3)**, (2005) 416-418

[8] Y. Song, L. Qi, Analy. Appl. **19(5)**, (2021) 779-800.

[9] L. Qi, Y. Song, X. Zhang, J. Nonlinear Convex Anal. **23(2)**, (2022) 191-213